บทความวิจัย

การสร้างตัวแบบการเจริญเติบโตของฟาสท์แพลนท์โดยใช้ริชาร์ดฟังก์ชัน

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บทคัดย่อ

ในงานนี้มีการใช้ริชาร์ดฟังก์ชันในรูป $y(t) = A(1+be^{-kt})^{1/(1-m)}$ มาวิเคราะห์การเจริญเติบโตของฟาสท์แพลนท์อย่างละเอียด มีการสร้างสูตรของอนุพันธ์อันดับหนึ่ง อันดับสอง และอันดับสามของฟังก์ชันดังกล่าวนี้เพื่อนำไปสู่การสร้างพารามิเตอร์ในรูป $G_{1,2} = f(m)$ ที่สามารถใช้คำนวณหาจุดวิกฤตสำคัญซึ่งแสดงถึงระยะของการเจริญเติบโตแต่ละระยะของฟาสท์แพลนท์ได้ มีการคำนวณค่าสัมประสิทธิ์ทั้งหมดของริชาร์ดฟังก์ชันซึ่งสร้างจากข้อมูลมวลแห้งสะสมของฟาสท์แพลนท์ด้วยวิธีการเชิงตัวเลข มีการสร้างกราฟเส้นโค้งการเจริญเติบโต อัตราการเจริญเติบโต อัตราเร่งการเจริญเติบโต มีการทดสอบด้วยวิธีการทางสถิติเพื่อให้การประมาณค่าการเจริญเติบโตของฟาสท์แพลนท์ด้วยริชาร์ดฟังก์ชันสามารถนำไปใช้ได้ ผลลัพธ์ที่ได้มีผลต่อประสิทธิภาพของผลผลิตทางการเกษตรที่ดีขึ้น และเป็นการแสดงให้เห็นถึงประโยชน์ของการประยุกต์ใช้คณิตศาสตร์ทางด้านการเกษตร

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Research Article

Fast plant growth modeling by using Richards's function

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Abstract

In this paper the Richards's function in the form $y(t) = A\left(1 + be^{-kt}\right)^{1/(1-m)}$ was used to precisely analyze Fast plant growth. The first, second and third derivative formulae of above function were given. A new parameter $G_{1,2} = f(m)$ was derived, enabling the calculation of the coordinates of critical points which mark the principal growth phases. The coefficients of Richards's function describing the dry matter accumulation in Fast plants were numerically calculated. The growth curve, growth rate curve and the growth acceleration curve were also drawn. A high usefulness of approximation of the growth process of Fast plants by means of Richards's function was confirmed statistically. This will result in better agricultural productivity and this is the benefit of mathematical applications to agriculture.

Keywords: Plant growth analysis, Richards's function, Critical points of growth, Fast plant

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Introduction

Agricultural mathematics is very important because it is the way to apply mathematics in agriculture that provides important information to help agricultural process accomplished consistency with that objective. Plant growth curve model is a crucial one of agricultural mathematics that the result from its information is very useful to support efficiency in planting and harvesting process (Boote et al., 1996; Prusinkiewicz, 2004; Fourcaud, 2008; Archontoulis & Miquez, 2013). However, in Thailand, agricultural mathematics is pretty rare using in agriculture and is viewed as not important and not useful. This paper was the case study of fast plant growth modeling by using Richards's function. The process of the study included, Fast plants planting and dry matter gathering, formulating model based on data with non-linear regression, testing model with the goodness of fit test (Gregorczyk, 1998; Paine et al., 2012), applying first and second derivative to consider rate and accelerate of growth. The results of the study elucidated the powerful application of mathematics in agriculture because of information from interpretation of model.

Richards's function

In Gregorczyk (1998) had concluded about comprehensive Richards's function that has been often applied in the quantitative analysis of plant growth. Richard proposed the following growth model:

$$y(t) = A\left(1 + be^{-kt}\right)^{1/(1-m)} \quad \text{when } m > 1,\ b > 1 \qquad (1)$$

where $y =$ growth size (e.g. dry matter), $t =$ time, $A =$ asymptotic value of growth size, m, b, $k =$ constant coefficients ($k > 0$).

The Richards function is very flexible and has a horizontal asymptote, and its graph has a characteristic sigmoid shape. Equation (1) can be transformed to a linear form:

$$\ln\left[(y/A)^{1-m} - 1\right] = \ln b - kt \qquad (2)$$

The linear form of Richards function cannot be useful for determining its constant parameters using analytical methods. This is because the considered growth model has up to four independent coefficients: $A, b,\ k, m$. The constant coefficients of Richards function may be calculated (after previously estimating their values) using numerical methods.

During plant growth, there are three characteristic critical moments that can be marked as $P_1, P_I,\ P_2$ on the graphs. At the point P_1, maximum acceleration of growth and the first inflection of the growth rate curve take place. At the point P_I the growth rate attains its maximum point. At the point P_2, maximum deceleration takes place, i.e. maximum negative acceleration of growth, while at the same time there is the second inflection of the growth rate curve. The points P_1 and P_2 separate the entire period growth into three phases: the exponential phase, the linear phase (from P_1 and P_2) and the ageing phase. Because of the importance of these three critical points a parameter $G_{1,2} = f(m)$ was derived, enabling the calculation of the coordinates of critical points which mark the principal growth phases and it is important to know their coordinates on the appropriate graphs. To achieve this, to analyze the variation course of Richards function, it is necessary to calculate its first, second and third derivative with respect to time.

The first derivative of Richards's function (1) is equal to:

$$y' = dy/dt = ky/(m-1)\left[1-(y/A)^{m-1}\right] \quad (3)$$

or

$$y' = Aky/(m-1)e^{-kt}\left(1+be^{-kt}\right)^{m/(1-m)} \quad (4)$$

The second derivative is in the form:

$$d^2y/dt^2 = y'' = ky'/(m-1)\left[1-(y/A)^{m-1}\right] \quad (5)$$

$$y'' = \frac{Abk^2e^{-kt}}{(m-1)^2\left(1+be^{-kt}\right)}\left(1+be^{-kt}\right)^{m/(1-m)}\left[b+(1-m)e^{kt}\right] \quad (6)$$

The third derivative can be presented in the following expressions:

$$\frac{d^3y}{dx^3} = y''' = \frac{ky'}{m-1}\left\{\frac{k}{m-1}\left[1-m\left(\frac{y}{A}\right)^{m-1}\right]^2 - y'm(m-1)\frac{1}{A}\left(\frac{y}{A}\right)^{m-2}\right\} \quad (7)$$

$$y''' = \frac{ky'}{(m-1)^2}\left[1-\left(\frac{y}{A}\right)^{m-1}\right]\cdot \left\{\frac{k}{m-1}\left[1-m\left(\frac{y}{A}\right)^{m-1}\right]^2 - \frac{k}{m-1}m(m-1)\left[1-\left(\frac{y}{A}\right)^{m-1}\right]\left(\frac{y}{A}\right)^{m-1}\right\} \quad (8)$$

The first derivative is positive, so function is always ascending; it also has horizontal asymptote $y = A$.

$$\lim_{t\to\infty} A\left(1+be^{-kt}\right)^{1/(1-m)} = A \quad (9)$$

The relative growth rate RGR amounts to:

$$\text{RGR} = y'/y = k/(m-1)\left[1-(y/A)^{m-1}\right] \quad (10)$$

By the nature of plant growth curve behavior, equating second derivative to zero we obtain coordinates of the inflection point $P_I(t_I, y_I)$ of Richards curve:

$$t_I = (1/k)\ln\left[b/(m-1)\right], \quad y_I = Am^{1/(1-m)} \quad (11)$$

The initial value of growth curve for $t = 0$ amounts to:

$$y(0) = A(1+b)^{1/(1-m)} \quad (12)$$

The coordinates of inflection points of the growth rate curve are obtained after equating the third derivative to zero. After substitution in equation (8):

$$(y/A)^{m-1} = x \quad (13)$$

a square equation is obtained:

$$(2m^2 - m)x^2 - \left(m^2 + m\right)x + 1 = 0 \quad (14)$$

Equation (14) has two roots:

$$x_{1,2} = \left[\frac{m(m-1) \pm (m-1)\sqrt{m(m+4)}}{2m(2m-1)}\right]^{1/(m-1)} \quad (15)$$

It seems as if the intentional introduction of new dual coefficient $G_{1,2}$, facilitates calculations:

$$G_{1,2} = \left[\frac{m(m+1) \mp (m-1)\sqrt{m(m+4)}}{2m(2m-1)} \right]^{1/(m-1)} \quad (16)$$

Now using equation (13) we can give the ordinates of two remaining critical of analyzed growth curve:

$$y_1 = AG_1 \quad , \quad y_2 = AG_2 \quad (17)$$

The non-dimensional coefficients G_1 and G_2 are included in interval $(0,1)$ and are defined, respectively, as a part of the final size attained by the plants in critical points P_1 and P_2.

Using equations (2), (11) and (17), abscissas of the inflection points of the growth rate curve are obtained in the form:

$$t_1 = t_I + \frac{1}{k} \ln \left[\frac{(m-1)G_1^{m-1}}{1-G_1^{m-1}} \right] \quad (18)$$

$$t_2 = t_I + \frac{1}{k} \ln \left[\frac{(m-1)G_2^{m-1}}{1-G_2^{m-1}} \right] \quad (19)$$

With formulae (18) and (19) we can calculate the time duration of the linear phase Δt:

$$\Delta t = t_2 - t_1 - = \frac{1}{k} \ln \frac{G_2^{m-1}\left(1-G_1^{m-1}\right)}{G_1^{m-1}\left(1-G_2^{m-1}\right)} \quad (20)$$

Fast Plant Growth Modeling

The above theory was used for the precise description of dry weight accumulation in Rapid Cycling in *Brassica rapa* Fast Plants. The fast plants grew in optimal condition of fertility and soil humidity in pots. The initial dry weight was assumed by seed dry weight and then, the dry matter of top soil parts was taken every 3 days after planting. The experiment was conducted at the stage of full maturity. Table 1 presents the mean values obtained from the measurements of 10 plants (5 pots with 2 plants each) and figure 1 presents some procedures in the experiment.

Table 1. Experimental and theoretical values of dry matter accumulation in the fast plants during vegetation

Time (days)	Values (g)	
	Experimental	Theoretical
0	0.0015	0.0010
3	0.0031	0.0034
6	0.0089	0.0101
9	0.0295	0.0268
12	0.0549	0.0605
15	0.1262	0.1131
18	0.1514	0.1760

Time (days)	Values (g)	
	Experimental	Theoretical
21	0.2699	0.2350
24	0.2431	0.2806
27	0.3386	0.3115
30	0.3172	0.3307
33	0.3605	0.3421
36	0.3418	0.3486
39	0.3128	0.3522
42	0.3864	0.3543

Random plants

Preparing to dry in oven

Drying

Measuring dry weight

Figure 1. Experimental processes

The primary data contained in table 1 helped in calculating the constant coefficients of Richards function. Coefficients A, b, k, m were numerically derived by using SPSS for Windows version 15. The starting values of the desired parameters were given based on the results of the preliminary approximation (Gregorczyk, 1998). After achieving precision in task calculation the theoretical model by Richards was obtained:

$$y = 0.357(1+13.642e^{-0.198t})^{1/(1-1.460)} \quad (21)$$

$$y' = \frac{(0.357)(13.642)(0.198)}{(0.46)} e^{-0.198t}(1+13.642e^{-0.198t})^{1.46/(1-1.460)} \quad (22)$$

$$y' = \frac{(0.357)(13.642)(0.198^2)e^{-0.198t}}{(0.46)^2(1+13.642e^{-0.198t})}(1+13.642e^{-0.198t})^{1.46/(1-1.460)}(1-0.46e^{0.198t}) \qquad (23)$$

Statistical analysis points to a high approximation efficacy in fast plant growth using the Richards function. It is confirmed by the high value of the determination coefficient $R^2 = 97.60$ %. The χ-square goodness-of-fit test was also applied. The empirical value was $\chi^2 = 0.0277$ at critical value $\chi^2_{df=14,\alpha=0.05} = 23.69$.

Table 2: Coordinates of the critical points of the growth curves of the Fast plants.

Point	Abscissa t (days)	y (g)	Ordinate y' (g day^{-1})	y'' (g day^{-2})
P_1	11.36	0.0514	0.0131	0.0023
P_I	17.12	0.1568	0.0213	0
P_2	22.80	0.2638	0.0148	-0.0017

Knowing the value of the coefficient m in Richards function enables the calculation of parameters $G_1 = 0.144$ and $G_2 = 0.743$, and the next coordinates of the critical points in the growth of fast plant. The values of the coordinates of these points are shown in Table 2.

Against a background of the experimental points, a course of the growth curve was drawn (figure 1). The estimated values of the dry matter of investigated plants are given in Table 1. The growth rate curve (figure 2.) and the growth acceleration curve (figure 3) were also drawn. The critical points (P_1, P_I, P_2) which characterize the critical moments during the growth of fast plant are marked in all the curves.

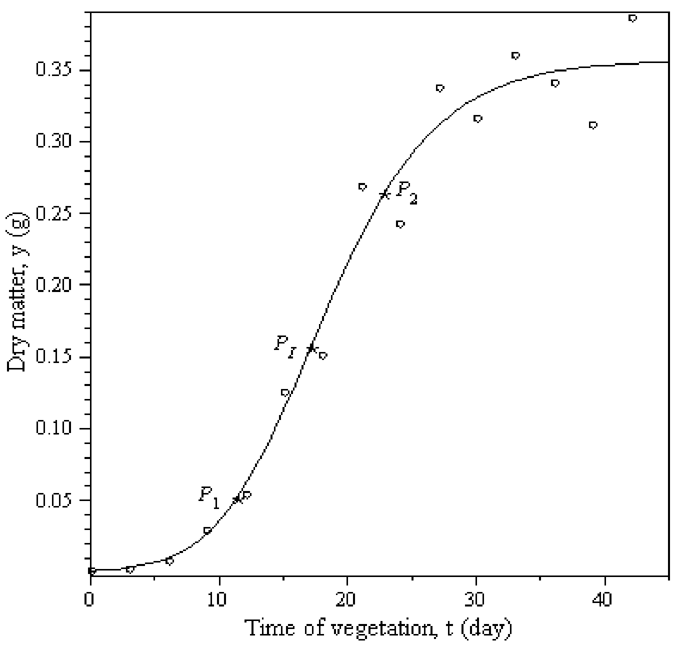

Figure 2. Richards growth curve of the Fast plants as compared with experimental points.

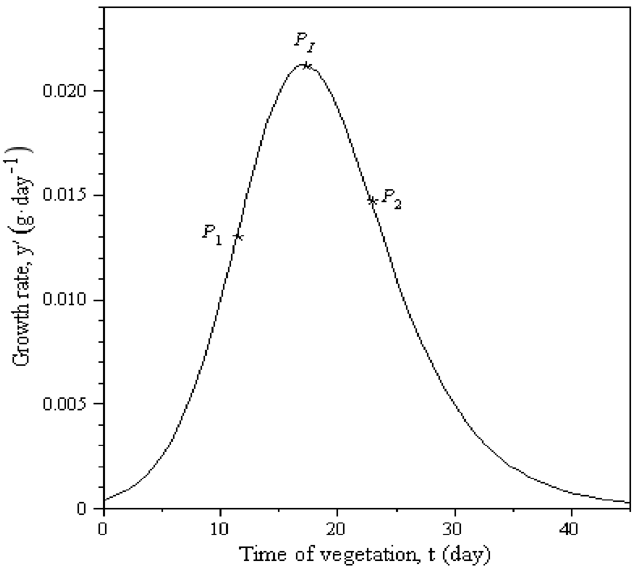

Figure 3. Growth rate curve of the Fast plants

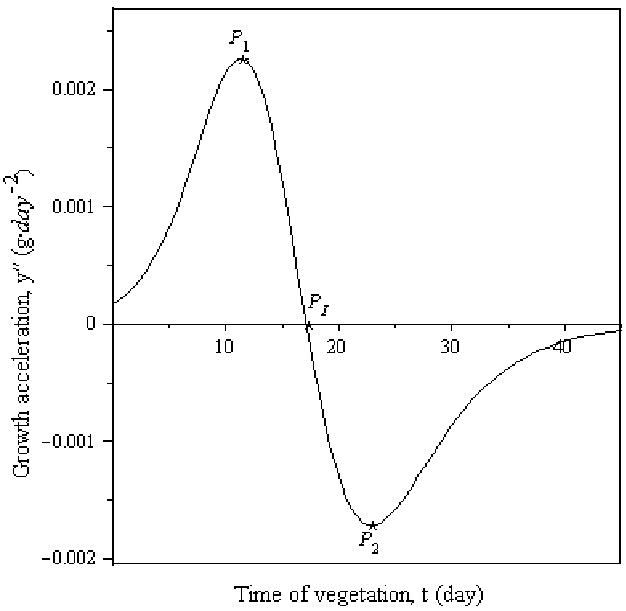

Figure 4. Growth acceleration curve of the Fast plants.

The top dry weight was initially small ($y_0 = 0.0010$ g), and increased exponentially to the beginning of the flowering stage (point P_1) on the 11th day of growth. At the moment, the acceleration of growth achieved its maximum value, $y_1'' = 0.0023$ g day^{-2} and the linear phase of growth starts, and lasted for about 12 days. At about the 17th day of growth (in the middle of flowering) the inflection of the curve occurred (point P_I) which simultaneously corresponded to maximum growth rate, $y_I' = 0.0213$ g day^{-1} and zero acceleration. The linear phase of growth ended at about the 23rd day of vegetation (point P_2). At this time the plants were at the early dough stage and growth acceleration had a minimum value of $y_2'' = -0.0017$ g day^{-2}. Later, there was an exponential fall in the growth rate of fast plant with relatively small increase in dry weight which lasted until the full maturity stage. Generally, it could be noticed that the accumulation curve of buck wheat dry matter had a typical sigmoid shape. In all the graphs, the three characteristic phases of growth, marked as critical points, could be precisely defined.

Conclusion

Based on the analysis of the results of the experiments by Richard's function make sense of the value of growth rate and the growth acceleration of fast plant. These data are particularly useful for applying to agriculture to set up a fertilizer or nutrient plant suitable for plant requirements. This will result in better agricultural productivity and this is the benefit of mathematical applications to agriculture.

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